1. **A synopsis of Hilbert space theory**

Below is a summary of Hilbert space theory that you find in more detail in the book of Akhiezer and Glazman. All the ideas are from this book.

Definition 1.1. Inner product space. *An inner product space $\mathcal{R}$ is a linear space endowed with a function $(\cdot,\cdot):\mathcal{R}\times\mathcal{R}\to\mathbb{C}$ which has the following properties*

a) $(f,\alpha g+\beta h)=\alpha(f,g)+\beta(f,h)$ *all* $f,g,h\in\mathcal{R}$ *and all* $\alpha, beta\in\mathbb{C}$*,*

b) $(f,g)=\overline{(g,f)}$ *all* $f,g\in\mathcal{R}$*,*

c) $(f,f)\geq 0$ *and* 0 *only if* $f=0$*.*

We set $\|f\|=\sqrt{(f,f)}$.

A simple consequence is

Lemma 1.2. Schwarz.s inequality and triangle inequality. *For every $f,g\in\mathcal{R}$ we have that*

$$|(f,g)|\leq\|f\|\|g\| \tag{1}$$

and

$$\|f+g\|\leq\|f\|+\|g\| \ . \tag{2}$$

Proof. We may assume that neither f nor g are zero and we may also assume that $(f,g)\neq 0$ for otherwise the inequality (1) is obvious. Set

$$X=\frac{f}{\|f\|}\ ,\ Y=\frac{g}{|g\|}$$

and

$$e^{i\phi}=\frac{(X,Y)}{|(X,Y)|}\ .$$

The inequality (1) is equivalent to

$$|(X,Y)|\leq 1\ .$$

Now

$$0\leq\|e^{i\phi}X+Y\|^2=2+e^{-i\phi}(X,Y)+e^{i\phi}(Y,X)=2+2|(X,Y)|$$

and likewise

$$0\leq\|e^{i\phi}X-Y\|^2=2-e^{-i\phi}(X,Y)-e^{i\phi}(Y,X)=2-2|(X,Y)|$$

from which the result follows. The triangle inequality (2) is now a simple consequence of Schwarz's inequality, because

$$\|f+g\|^2=\|f\|^2+\|g\|^2+(f,g)+(g,f)\leq\|f\|^2+\|g\|^2+2\|f\|\|g\|=(\|f\|+\|g\|)^2\ .$$

□

Lemma 1.3. Parallelogram identity. *For any two vectors $f,g\in\mathcal{R}$ we have that*

$$\|f+g\|^2+\|f-g\|^2=2\|f\|^2+2\|g\|^2\ .$$

Proof. A simple computation. □

Remark 1.4. *Note that $\mathcal{R}$ endowed with the norm $\|\cdot\|$ is a metric space. The distance between two vectors f, g is given by $\|f - g\|$. Hence we know what the open and closed sets are. A set $S \subset \mathcal{R}$ is open if for any $f \in S$ there exists $\varepsilon > 0$ so that the ball*

$$B_\varepsilon(f) = \{g \in \mathcal{R} : \|f - g\| < \varepsilon\}$$

is a subset of S. A subset $T \subset \mathcal{R}$ is closed if its complement in $\mathcal{R}$, T^c is open. The following statement is easy to prove: A subset $T \subset \mathcal{R}$ is closed if and only if T contains all its limit points. More precisely, if $f^{(k)}$ is any sequence in T which converges to some element $f \in \mathcal{R}$, then $f \in T$.

Definition 1.5. Completeness. *A inner product space is complete if for every Cauchy Sequence $f^{(k)} \in \mathcal{R}$ there exists $f \in \mathcal{R}$ such that $f^{(k)}$ converges to f.*

Recall that the statement "$f^{(k)}$ converges to f" means that for any $\varepsilon > 0$ there exists a positive integer N so that for all $k > N$, $\|f^{(k)} - f\| < \varepsilon$. Also recall that the statement "$f^{(k)}$ is a Cauchy Sequence" means that for any $\varepsilon > 0$ there exists a positive integer N so that $\|f^{(k)} - f^{(\ell)}\| < \varepsilon$ for all $k, \ell > N$.

Definition 1.6. Hilbert space. *A complete inner product space is called a "Hilbert" space and we denote it by $\mathcal{H}$.*

A set $C \subset \mathcal{H}$ is convex, if with $f, g \in C$, $\lambda f + (1-\lambda)g \in C$ for all $0 \le \lambda \le 1$.

Theorem 1.7. Minimial distance. *Let C be a closed convex subset of a Hilbert space $\mathcal{H}$ and $f \in \mathcal{H}$ arbitrary. There exists a unique vector $h \in C$ such that*

$$\|f - h\| = d := \inf\{\|f - g\| : g \in C\} \ .$$

Proof. By the definition of the infimum, for any positive integer n there exists $g^{(n)} \in C$ such that

$$d \le \|f - g^{(n)}\| \le d + \frac{1}{n} \ .$$

Now for any positive integer n, m, by the parallelogram identity,

$$\|\frac{f-g^{(n)}}{2} + \frac{f-g^{(m)}}{2}\|^2 + \|\frac{f-g^{(n)}}{2} - \frac{f-g^{(m)}}{2}\|^2 = 2\|\frac{f-g^{(n)}}{2}\|^2 + 2\|\frac{f-g^{(m)}}{2}\|^2 \ ,$$

which can be rewritten as

$$\|f - \frac{g^{(n)}+g^{(m)}}{2}\|^2 + \|\frac{g^{(n)}-g^{(m)}}{2}\|^2 = 2\|\frac{f-g^{(n)}}{2}\|^2 + 2\|\frac{f-g^{(m)}}{2}\|^2 \ ,$$

Since C is convex, $\frac{g^{(n)}+g^{(m)}}{2} \in C$ and

$$d^2 \le \|f - \frac{g^{(n)}+g^{(m)}}{2}\|^2 + \|\frac{g^{(n)}-g^{(m)}}{2}\|^2 \le \frac{1}{2}(d+\frac{1}{n})^2 + \frac{1}{2}(d+\frac{1}{m})^2$$

or

$$\|\frac{g^{(n)}-g^{(m)}}{2}\|^2 \le d(\frac{1}{n}+\frac{1}{m}) + \frac{1}{2n^2} + \frac{1}{2m^2} \ .$$

Pick a positive integer N so that

$$4d(\frac{1}{n}+\frac{1}{m}) + \frac{2}{n^2} + \frac{2}{m^2} < \varepsilon^2$$

and we learn that whenever $n, m > N$,

$$\|g^{(n)} - g^{(m)}\| < \varepsilon \ .$$

Hence, $g^{(n)}$ is a Cauchy Sequence and since $\mathcal{H}$ is complete, there exists $h \in \mathcal{H}$ so that $g^{(n)}$ converges to h. Because C is closed, $h \in C$. We have to show that $\|f - h\| = d$. This follows easily, since

$$d \le \|f - h\| \le \|f - g^{(n)}\| + \|g^{(n)} - h\| \le d + \frac{1}{n} + \|g^{(n)} - h\| \ .$$

Suppose now that $h_1, h_2 \in C$ are two vectors with

$$d = \|f - h_1\| = \|f - h_2\| \ .$$

Then, again by the parallelogram identity

$$\|f - \frac{h_1 + h_2}{2}\|^2 + \|\frac{h_1 - h_2}{2}\|^2 = 2\|\frac{f - h_1}{2}\|^2 + 2\|\frac{f - h_2}{2}\|^2 = d^2 \ .$$

Hence

$$\|\frac{h_1 - h_2}{2}\|^2 = d^2 - \|f - \frac{h_1 + h_2}{2}\|^2 \le 0$$

since $\frac{h_1+h_2}{2} \in C$. □

Definition 1.8. Linear manifold, subspace *A subset $M \subset \mathcal{H}$ is a linear manifold if it is closed under addition of vectors and scalar multiplication. If a linear manifold $G \subset \mathcal{H}$ is closed, then it is a complete inner product space, i.e., a Hilbert space. In this case we call G a subspace of $\mathcal{H}$.*

We say that two vectors g, h are **orthogonal or perpendicular** to each other if $(f, g) = 0$.

Theorem 1.9. Orthogonal complement *Let G be a subspace of $\mathcal{H}$. Then for any $f \in \mathcal{H}$ there exits two uniquely specified vectors g and h such that $f = g + h$, $g \in G$ and $h \perp G$, i.e., h is perpendicular to every vector in G.*

Proof. The subspace G is convex (since it is linear) and closed. Hence there exists $g \in G$ so that

$$\|f - g\| = \inf\{\|f - u\| : u \in G\} \ .$$

We show that $f - g$ is perpendicular to every vector in G. Pick $v \in G$ and consider the vector $g + tv$ where t is an arbitrary complex number. Since $g + tv \in G$ we have that

$$d^2 \le |f - g - tv\|^2 = \|f - g\|^2 + \bar{t}(f - g, u) + t(u, f - g) + |t|^2\|u\|^2$$
$$= d^2 + \bar{t}(f - g, u) + t(u, f - g) + |t|^2\|u\|^2$$

Hence we have for all $t \in \mathbb{C}$

$$t(f - g, u) + \bar{t}(u, f - g) + |t|^2\|u\|^2 \ge 0 \ .$$

Choosing t real positive we have that

$$\Re(f - g, u) + t\|u\|^2 \ge 0$$

and since $t > 0$ is arbitrary we find $\Re(f - g, u) \ge 0$. Likewise, choosing $t < 0$ we find that $\Re(f - g, u) \le 0$ and hence $\Re(f - g, u) = 0$. Next we choose t to be purely imaginary and find, by a similar reasoning that $\Im(f - g, u) = 0$. Thus $(f - g, u) = 0$ and since $u \in G$ is arbitrary $f - g \perp G$. If $f = g_1 + h_1 = g_2 + h_2$ are two such decomposition, we have that

$$g_1 - g_2 = h_2 - h_1 \ .$$

Because $g_1 - g_2 \in G$ and $h_2 - h_1 \perp G$ we must have that $g_1 - g_2 = 0$ and hence $h_1 = h_2$. □

Definition 1.10. Bounded linear functionals *A bounded linear function is a function* $\ell : \mathcal{H} \to \mathbb{C}$ *that satisifes* $\ell(f+g) = \ell(f) + \ell(g)$ *for all* $f, g \in \mathcal{H}$ *and* $\ell(\alpha f) = \alpha \ell(f)$ *all* $f \in \mathcal{H}$ *and* $\alpha \in \mathbb{C}$. *Bounded means that there exists a positive constant* C *such that*

$$|\ell(f)| \leq C\|f\|$$

for all $f \in \mathcal{H}$.

Proposition 1.11. *A linear functional is bounded if and only if it is continuous.*

Proof. If ℓ is bounded then for any $f, g \in \mathcal{H}$

$$|\ell(f) - \ell(g)| = |\ell(f-g)| \leq C\|f-g\|$$

from which the continuity is an immediate consequence. Suppose now that ℓ is continuous. This mean that for any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$|\ell(f) - \ell(g)| < \varepsilon$$

whenever $\|f-g\| < \delta$. Hence, whenever $\|f\| < \delta$ then

$$|\ell(f)| < \varepsilon \ .$$

Now, pick any $f \in \mathcal{H}$ and consider

$$g = \frac{\delta}{2}\frac{f}{\|f\|} \ .$$

Then $\|g\| = \frac{\delta}{2} < \delta$ and

$$\ell(f) = \ell(g)\frac{2\|f\|}{\delta}$$

and hence

$$|\ell(f)| < \frac{2\varepsilon}{\delta}\|f\| \ .$$

□

Theorem 1.12. Riesz representation theorem *For any bounded linear functional* ℓ *on a Hilbert space* $\mathcal{H}$ *there exists a unique vector* $v \in \mathcal{H}$ *so that*

$$\ell(f) = (v, f)$$

for all $f \in \mathcal{H}$.

Proof. Consider the set

$$G = \{f \in \mathcal{H} : \ell(f) = 0\}$$

This set is a subspace G of the Hilbert space $\mathcal{H}$. Either the function ℓ is the trivial functional in which case $v = 0$ or there exists $g \in \mathcal{H}$ with $\ell(g) \neq 0$. By the projection lemma there exists a unique $u \perp G$ and $w \in G$ such that $g = u + w$ and $\ell(u) = \ell(g) \neq 0$. For $f \in \mathcal{H}$ arbitrary we have that $\ell(f)u - \ell(u)f \in G$ and by taking the inner product with u we find

$$\ell(f)(u,u) - \ell(u)(u,f) = 0$$

or

$$\ell(f) = \frac{\ell(u)}{(u,u)}(u,f)$$

and if we set

$$v = \frac{\overline{\ell(u)}}{(u,u)}u$$

we have that

$$\ell(f) = (v, f) \text{ for all } f \in \mathcal{H} \ .$$

If v_1, v_2 are two such functions then

$$(v_1 - v_2, f) = 0$$

for all $f \in \mathcal{H}$ and hence $v_1 = v_2$. □

We continue with an interesting topic that runs under the name "uniform boundedness principle". We deal with this in a different fashion than usual.

Consider a function

$$p : \mathcal{H} \to \mathbb{R} \ .$$

Recall tat p is continuous at $f_0 \in \mathcal{H}$ if for any $\varepsilon > 0$ there exist δ such that

$$\varepsilon > p(f) - p(f_0) > -\varepsilon$$

whenever $\|f - f_0\| < \delta$.

Note that the above definition has two parts: We say that p is **lower semi continuous** if for every $\varepsilon > 0$ there exists $\delta > 0$ so that

$$p(f) - p(f_0) > -\varepsilon$$

whenever $\|f - f_0\| < \delta$. It is **upper semi continuous** if

$$\varepsilon > p(f) - p(f_0)$$

whenever $\|f - f_0\| < \delta$.

It is easy to see that p is lower semi continuous if the set $\{f \in \mathcal{H} : p(f) > t\}$ is open for any $t \in \mathbb{R}$. Likewise, p is upper semi continuous if for all $t \in \mathbb{R}$ the set $\{f \in \mathcal{H} : p(f) < t\}$. In general, there is no reason why a lower semi continuous function is continuous but there is an interesting exception.

Definition 1.13. *Subadditive functions A function $p : \mathcal{H} \to \mathbb{R}$ is a* **seminorm** *if*

a) $p(f + g) \leq p(f) + p(g)$ all $f, g \in \mathcal{H}$

b) $p(\alpha f) = |\alpha| p(f)$.

Clearly, $p(0) = 0$ and $p(-f) = f$. Hence, $p(f) \geq 0$ for all $f \in \mathcal{H}$ since

$$0 = p(0) = p(f - f) \leq 2p(f) \ .$$

Note that $p(f)$ is almost a norm, except that $p(f) = 0$ does not entail that $f = 0$. Note, that the book calls such a function 'convex', which in its usual use has a different definition.

The following theorem is now of great interest. It is crucial that $\mathcal{H}$ is complete.

Theorem 1.14. *Any lower semi continuous seminorm $p : \mathcal{H} \to \mathbb{R}$ is bounded, i.e., there is a constant M such that*

$$p(f) \leq M\|f\| \ .$$

Proof. Given an open ball in the Hilbert space, $B_\rho(f)$. We shall prove that p is bounded on $B_\rho(f)$ if an only if it is bounded on the open unit ball, $B_1(0)$ centered at the origin. Assume that p is bounded on $B_1(0)$, i.e., $p(g) \le M$ for all g with $\|g\| < 1$. Pick any $h \in B_\rho(f)$. This means that $\|f-h\| < \rho$ or

$$\|\frac{f-h}{\rho}\| < 1 \ .$$

Hence

$$M \ge p(\frac{f-h}{\rho}) = \frac{1}{\rho} p(f-h)$$

and for all $h \in B_\rho(f)$

$$p(h) = p(h-f+f) \le p(h-f) + p(f) \le M\rho + p(f) \ .$$

Conversely, if $p(h) \le C$ for all $h \in B_\rho(f)$, then for $g \in B_1(0)$ we can write

$$g = \frac{h-f}{\rho}$$

where $f = h - g\rho$. Now

$$p(g) = p(\frac{h-f}{\rho}) = \frac{p(f-h)}{\rho} \le \frac{p(f)+p(h)}{\rho} \le \frac{2C}{\rho} \ .$$

Assume that p is not bounded in $B_1(0)$. Hence it is unbounded on every ball. There exists $f_1 \in B_1(0)$ such that

$$p(f_1) > 1 \ .$$

Since p is lower semi continuous, there exists $\rho_1 < 1/2$ such that $p(f) > 1$ on the ball $B_{\rho_1}(f_1)$. Since p is not bounded on $B_{\rho_1}(f_1)$, there exists $f_2 \in B_{\rho_1}(f_1)$ with $p(f_2) > 2$. Again, since p is lower semi continuous, there exists $\rho_2 < \rho_1/2$ so that $p(f) > 2$ on all of $B_{\rho_2}(f_2)$. Continuing in this fashion we have a sequence of balls

$$B_{\rho_1}(f_1) \supset B_{\rho_2}(f_2) \supset B_{\rho_3}(f_3) \cdots$$

with $\rho_k < \rho_{k-1}/2$ so that $p(f) > k$ for all $f \in B_{\rho_k}(f_k)$. Since f_k is a Cauchy Sequence and $\mathcal{H}$ is complete, there exists f with $\lim_{k\to\infty} f_k = f$. Further, since $f \in B_{\rho_k}(f_k)$ for all k we have that $p(f) > k$ for all k which is not possible.

□

Remark 1.15. *Let us look at the following 'argument' that does not use lower semi continuity: Assume that p is not bounded in $B_1(0)$. Hence it is unbounded on every ball. There exists $f_1 \in B_1(0)$ such that*

$$p(f_1) > 1 \ .$$

Pick $\rho_1 < 1/2$ so that $B_{\rho_1}(f_1) \subset B_1(0)$. (Note, that we do not know whether $p(f) > 1$ for all $f \in B_{\rho_1}(f_1)$.)Since p is not bounded on $B_{\rho_1}(f_1)$ there exists $f_2 \in B_{\rho_1}(f_1)$ with $p(f_2) > 2$. Pick $\rho_2 < \rho_1/2$ There exists $f_3 \in B_{\rho_2}(f_2)$ with $p(f_3) > 3$ and so on. We then have

$$B_{\rho_1}(f_1) \supset B_{\rho_2}(f_2) \supset B_{\rho_3}(f_3) \cdots$$

and $p(f_k) > k$. The sequence f_k is a Cauchy sequence and hence converges to some f. Note that we cannot say anything about the value of $p(f)$.